# Notes on Green's Functions for Nonhomogeneous Equations*

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The Green's function method is a powerful method for solving nonhomogeneous linear equations $Ly(x) = f(x)$. The Green's function $G(x, a)$ satisfies the equation

$$LG(x, a) = \delta(x - a),$$

where $\delta$ denotes the Dirac delta function.

**Learning objectives**

- Delta functions, Heaviside functions

- Use of Green's functions for solving nonhomogeneous equations

- Green's functions vs. variation of parameters

## 1    Dirac Delta Function and Heaviside Function

Definition.

$$\begin{cases} \delta(x - a) = 0 \text{ if } x \neq a \\ \int_{-\infty}^{+\infty} \delta(x - a) dx = 1 \end{cases}$$

Alternatively,

$$\delta(x - a) = \lim_{\varepsilon \to 0+} F_\varepsilon(x),$$

where

$$F_\varepsilon(x) = \begin{cases} 0 & x < a - \frac{1}{2}\varepsilon \\ 1/\varepsilon & a - \frac{1}{2}\varepsilon \leq x \leq a + \frac{1}{2}\varepsilon \\ 0 & x > a + \frac{1}{2}\varepsilon. \end{cases}$$

Note that area of $F_\varepsilon(x)$ is always 1. The delta function can also be defined as

$$\delta(x - a) = \lim_{\varepsilon \to 0+} \frac{\varepsilon}{\pi[(x - a)^2 + \varepsilon^2]};$$

$$\delta(x - a) = \lim_{\varepsilon \to 0+} \frac{e^{-(x-a)^2/\varepsilon}}{\sqrt{\pi \varepsilon}}$$

$$\delta(x - a) = \lim_{L \to +\infty} \frac{1}{2\pi} \int_{-L}^{L} e^{i(x-a)t} dt.$$

Alternatively, $\delta(x - a)$ can be defined as the derivative of the Heaviside function $H(x - a)$,

$$H(x - a) = \begin{cases} 0 & x < a \\ 1/2 & x = a \\ 1 & x > a \end{cases}.$$

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Then
$$\delta(x - a) = \frac{d}{dx}H(x - a).$$

Note that Heaviside is "smoother" than the Dirac delta function, as integration is a smoothing operation. Furthermore, the integral of the Heaviside function is a ramp function

$$R(x - a) = \int_{-\infty}^{x} H(t - a)dt = \begin{cases} 0 & x \le a \\ x - a & x > a \end{cases}.$$

The most important property of Dirac delta function we need is that

$$\int_{-\infty}^{\infty} \delta(x - a)f(x)dx = \int_{-\infty}^{\infty} \delta(x - a)f(a)dx$$
$$= f(a) \int_{-\infty}^{\infty} \delta(x - a)dx$$
$$= f(a).$$

# 2    Application to Differential Equations

The Green's function $G(x, a)$ associated with the nonhomogeneous equation $Ly = f(x)$ satisfies the differential equation
$$LG(x, a) = \delta(x - a).$$
Once $G(x, a)$ is known, then the solution to $Ly = f(x)$ is

$$y(x) = \int_{-\infty}^{\infty} f(a)G(x, a)da,$$

because

$$\begin{aligned} Ly(x) &= L \int_{-\infty}^{\infty} f(a)G(x, a)da \\ &= \int_{-\infty}^{\infty} f(a)\left(LG(x, a)\right)da \\ &= \int_{-\infty}^{\infty} f(a)\delta(x - a)da \\ &= \int_{-\infty}^{\infty} f(x)\delta(a - x)da \\ &= f(x) \int_{-\infty}^{\infty} \delta(a - x)da \\ &= f(x). \end{aligned}$$

In the second equality, the $L$-operator and the integration can be interchanged because they are on two different variables. Also note that it is in effect also true for

$$y(x) = \int_{I} f(a)G(x, a)da,$$

over any interval $I$ that contains $x$ on which $f(x)$ is continuous, instead of just over $(-\infty, \infty)$.

The question now is how to solve the Green's function problem. It is easy to solve if the solutions of the homogeneous system $Ly(x) = 0$ are known. We illustrate with the 2nd order equation with

$$L = \frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_0(x).$$

$$LG(x,a) = \left[\frac{d^2}{dx^2} + p_1(x)\frac{d}{dx} + p_0(x)\right]G(x,a) = \delta(x-a).$$

Assume $c_1 y_1 + c_2 y_2$ is the general solution of $Ly = 0$. When $x \neq a$, the RHS vanishes, so

$$G(x,a) = A_1 y_1(x) + A_2 y_2(x), \; x < a$$
$$G(x,a) = B_1 y_1(x) + B_2 y_2(x), \; x > a.$$

There are two extra conditions:

1. $G(x,a)$ is continuous at $x = a$.

2. $\partial G/\partial x$ has a finite jump discontinuity of magnitude 1 at $x = a$.

To prove condition 1, note that if $G(x,a)$ is discontinuous, then $LG(x)$ would be even more singular than the $\delta$ function.

For condition 2, first $\partial G/\partial x$ must have a finite jump due to condition 1. From $LG(x,a) = \delta(x-a)$, we see that

$$\partial^2 G/\partial x^2 - \delta(x-a)$$

has a finite jump, and its integral

$$\int \partial^2 G/\partial x^2 \text{-} \delta(\text{x-a})dx = \partial G/\partial x - H(x,a)$$

must be continuous everywhere, so $\partial G/\partial x$ has the same jump of 1 as $H(x,a)$ at $a$, i.e.

$$\lim_{\varepsilon \to 0+} \left[\frac{\partial G}{\partial x}\Big|_{x=a+\varepsilon} - \frac{\partial G}{\partial x}\Big|_{x=a-\varepsilon}\right] = 1.$$

We can further verify it as follows. Let $\int_{a-0}^{a+0} \equiv \lim_{\varepsilon \to 0}\int_{a-\varepsilon}^{a+\varepsilon}$.

$$\int_{-\infty}^{+\infty} LG(x,a) = \int_{a-0}^{a+0}\frac{d^2}{dx^2}G(x,a)dx + \underbrace{\int_{a-0}^{a+0} p_1(x)\frac{d}{dx}G(x,a)dx}_{\int_{a-0}^{a+0}p_1 H(x-a)dx=0} + \underbrace{\int_{a-0}^{a+0} p_0(x)G(x,a)dx}_{0}$$

$$= \int_{a-0}^{a+0}\frac{dH(x-a)}{dx}dx$$

$$= \int_{a-0}^{a+0}\delta(x-a)dx.$$

The continuity of $G(x,a)$ at $x = a$ gives the condition

$$A_1 y_1(a) + A_2 y_2(a) = B_1 y_1(a) + B_2 y_2(a)$$

and the jump of $\partial G(x,a)/\partial x$ gives the equation

$$B_1 y_1'(a) + B_2 y_2'(a) - A_1 y_1'(a) - A_2 y_2'(a) = 1.$$

Using these equations, we obtain

$$B_1 - A_1 = -\frac{y_2(a)}{W[y_1(a), y_2(a)]},$$
$$B_2 - A_2 = \frac{y_1(a)}{W[y_1(a), y_2(a)]}.$$

We have two equations for four constants. $LG = \delta(x-a)$ has no unique because $G + y_c$ is also a solution, and therefore there are two free parameters. These parameters may be determined by initial conditions or boundary conditions.

Here for simplicity, we can just assume $A_1 = A_2 = 0$, then

$$G(x, a) = \begin{cases} 0 & x < a \\ \frac{-y_2(a)y_1(x) + y_1(a)y_2(x)}{W[y_1(a), y_2(a)]} & x \geq a. \end{cases}$$

The solution to $Ly = f$ is then

$$
\begin{aligned}
y(x) &= \int_{-\infty}^{\infty} f(a)G(x, a)da \\
&= -y_1(x) \int_{-\infty}^{x} \frac{f(a)y_2(a)}{W[y_1(a), y_2(a)]}da + y_2(x) \int_{-\infty}^{x} \frac{f(a)y_1(a)}{W[y_1(a), y_2(a)]}da
\end{aligned}
$$

This is identical to the solution obtained using variation of parameters.

Note that the choice of $A_1 = A_2 = 0$ was arbitrary. We could have chosen $B_1 = B_2 = 0$, and then obtain

$$G(x, a) = \begin{cases} \frac{y_2(a)y_1(x) - y_1(a)y_2(x)}{W[y_1(a), y_2(a)]} & x < a \\ 0 & x \geq a \end{cases}$$

and

$$
\begin{aligned}
y(x) &= \int_{-\infty}^{\infty} f(a)G(x, a)da \\
&= y_1(x) \int_{x}^{\infty} \frac{f(a)y_2(a)}{W[y_1(a), y_2(a)]}da - y_2(x) \int_{x}^{\infty} \frac{f(a)y_1(a)}{W[y_1(a), y_2(a)]}da.
\end{aligned}
$$

In addition, the limits $(-\infty, \infty)$ of the integral can be replaced by other nonempty interval $I$, and the resulting $y(x)$ would still be a particular solution of the nonhomogeneous equation.

**Advantages of Green's Function Approach** The Green's function approach is particularly better to solve boundary-value problems, especially when $L$ and the boundary conditions are fixed but the RHS may vary. It is easy for solving boundary value problem with homogeneous boundary conditions. For nonhomogeneous boundary conditions for which the BVP has solutions, some transformations of the variable are needed to homogenize the boundary conditions.

**Example 1** Consider the boundary value problem

$$y'' = f(x), \quad \text{with } y(0) = 0 \text{ and } y'(1) = 0.$$

The Green's function problem is

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2}G(x, a) &= \delta(x - a) \\
G(0, a) &= 0 \\
\frac{\partial}{\partial x}G(1, a) &= 0,
\end{aligned}
$$

for which, the solution is

$$y(x) = \int_0^1 G(x, a)f(a)da,$$

since

$$
\begin{aligned}
y(0) &= \int_0^1 G(0, a)f(a)da = \int_0^1 0 f(a)da = 0 \\
y'(1) &= \int_0^1 \frac{\partial}{\partial x}G(1, a)f(a)da = \int_0^1 0 f(a)da = 0.
\end{aligned}
$$

If $x \neq a$, $\frac{d^2}{dx^2}G = 0$, so

$$G = \begin{cases} A_1x + A_2 & \text{if } x < a \\ B_1x + B_2 & \text{if } x > a \end{cases},$$

where $a \in (0,1)$. Note that

$$G(0,a) = 0 \quad \Rightarrow \quad A_2 = 0 \Rightarrow G(x,a) = A_1x \text{ for } x < a$$
$$G'(0,a) = 0 \quad \Rightarrow \quad B_1 = 0 \Rightarrow G(x,a) = B_2 \text{ for } x > a.$$

From the continuity of $G$ and the jump of $\partial G/\partial x$ at $x = a$, we have

$$A_1a = B_2$$
$$0 - A_1 = 1.$$

Therefore, $A_1 = -1$ and $B_2 = -a$, and

$$G(x,a) = \begin{cases} -x & \text{if } x < a \\ -a & \text{if } x \geq a \end{cases}.$$

The solution is $y(x) = \int_0^1 G(x,a)f(a)da$. Suppose $f(x) = x^2$. Then

$$
\begin{aligned}
y(x) &= \int_0^x (-a)\, a^2 da + \int_x^1 (-x)a^2 da \\
&= -\frac{a^4}{4}\Big|_0^x - x\frac{a^3}{3}\Big|_x^1 \\
&= -\frac{x^4}{4} - \frac{1}{3}x + \frac{x^4}{3} \\
&= \frac{x^4}{12} - \frac{x}{3}.
\end{aligned}
$$

**Example 2**   Consider the BVP

$$y'' - y = f(x), \quad y(\pm\infty) = 0.$$

The Green's function problem is

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2}G(x,a) - G(x,a) &= \delta(x-a) \\
G(-\infty,a) &= 0 \\
G(\infty,a) &= 0,
\end{aligned}
$$

for which, the solution is

$$y(x) = \int_{-\infty}^{\infty} G(x,a)f(a)da,$$

since

$$
\begin{aligned}
y(-\infty) &= \int_{-\infty}^{\infty} G(-\infty,a)f(a)da = 0 \\
y(\infty) &= \int_{-\infty}^{\infty} G(\infty,a)f(a)da = 0.
\end{aligned}
$$

The homogeneous equation has solutions $y_1 = e^{-x}$ and $y_2 = e^x$. Therefore,

$$G(x,a) = \begin{cases} A_1e^{-x} + A_2e^x & \text{if } x < a \\ B_1e^{-x} + B_2e^x & \text{if } x > a \end{cases}.$$

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From the boundary conditions, we have $A_1 = 0$ and $B_2 = 0$.

The continuity and the jump discontinuity of derivatives at $x = a$ give

$$\begin{aligned} A_2 e^a &= B_1 e^{-a} \\ -B_1 e^{-a} &= A_2 e^a + 1. \end{aligned}$$

Solve the linear system, we obtain

$$B_1 = -\frac{1}{2} e^a, \quad A_2 = -\frac{1}{2} e^{-a}$$

and

$$G(x, a) = \begin{cases} -\frac{1}{2} e^{x-a} & \text{if } x < a \\ -\frac{1}{2} e^{a-x} & \text{if } x > a \end{cases} = -\frac{1}{2} e^{-|x-a|}.$$

Therefore,

$$y(x) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|x-a|} f(a) da.$$